COUNTING GROWTH TYPES OF AUTOMORPHISMS OF FREE GROUPS

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Abstract. Given an automorphism of a free group F_n, we consider the following invariants: e is the number of exponential strata (an upper bound for the number of different exponential growth rates of conjugacy classes); d is the maximal degree of polynomial growth of conjugacy classes; R is the rank of the fixed subgroup. We determine precisely which triples (e, d, R) may be realized by an automorphism of F_n. In particular, the inequality $e \leq \frac{3n-2}{4}$ (due to Levitt-Lustig) always holds. In an appendix, we show that any conjugacy class grows like a polynomial times an exponential under iteration of the automorphism.

Introduction

Consider an automorphism α of a free group F_n which is induced by a homeomorphism h of a compact surface Σ. After replacing h by a power, it becomes isotopic to a homeomorphism h' with a very simple structure: Σ is a union of invariant subsurfaces Σ_i, and on each subsurface h' is either pseudo-Anosov, or a Dehn twist in an annulus, or the identity.

Three very different behaviors thus appear: exponential, linear, trivial. This qualitative analysis also has a quantitative side. Just writing that the sum of the Euler characteristics of the Σ_i's is $\chi(\Sigma)$ gives bounds for, say, the rank of the fixed subgroup of α, or the number of subsurfaces.

If α is an arbitrary automorphism of F_n, there is no general analogue of the Nielsen-Thurston decomposition into invariant subsurfaces. Still, one can again distinguish three different behaviors. Exponential behavior comes from exponentially growing strata in a relative train track representative of α, or, more intrinsically, from the attracting laminations. Linear should now be replaced by polynomial: unlike in the surface case, it is possible for the length of a conjugacy class to grow as a polynomial of degree > 1 under iteration of α (the simplest example is the automorphism of F_3 which sends a to a, b to ba, c to cb, with c growing quadratically). Trivial behavior comes from the fixed subgroup of α.

Though these three different behaviors are not separated as clearly as in the surface case, the goal of this paper is to show that one can still give precise numerical bounds for invariants that measure how much "space" each of these three behaviors occupies within F_n.

The first invariant attached to an automorphism α is the number e of attracting laminations (equal to the number of exponential strata of an improved train track representative). It is an upper bound for the number of exponential growth rates of conjugacy classes (see appendix). In the surface case, e is the number of subsurfaces Σ_i on which the map is pseudo-Anosov (it is a pleasant exercise to compute the maximal value of e on a given orientable surface Σ; the answer is given in Remark 5.1).

The second invariant (which has no equivalent in the surface case) is the maximal degree d such that the length of some conjugacy class grows as a polynomial of degree d under iteration of α (it equals 2 in the example given above on F_3).

The third invariant is the rank $R = \operatorname{rk}\operatorname{Fix}\alpha$ of the fixed subgroup of α. It is a famous theorem by Bestvina-Handel [4] that R does not exceed n.

Our main result is a precise characterization of which triples (e, d, R) may be realized by an automorphism of F_n. Let us first consider e and d.

Theorem 1. *Given $\alpha \in Aut(F_n)$, the numbers e and d satisfy:*

$$\begin{aligned} e + d &\leq n - 1 \\ 4e + 2d &\leq 3n - 2 \qquad (\leq 3n - 3 \text{ if } d > 0). \end{aligned}$$

In particular, $e \leq \dfrac{3n-2}{4}$ (Levitt-Lustig, [11]).

Conversely, any (e, d) satisfying these inequalities may be realized by some $\alpha \in$ Aut (F_n).

The inequalities are equivalent to saying that (e, d) belongs to the closed quadrilateral with vertices $(0, 0), (0, n-1), (\frac{n-1}{2}, \frac{n-1}{2}), (\frac{3n-2}{4}, 0)$ pictured on Figure 1. The equality $e = \left[\frac{3n-2}{4}\right]$ is achieved by a surface homeomorphism with all subsurfaces Σ_i either once-punctured tori or four-punctured spheres (see Figure 2 for a picture when $n \equiv 2 \mod 4$); recall that there is no pseudo-Anosov map on a pair of pants.

Theorem 2. *Given e and d satisfying the conditions above, the possible values of* $\operatorname{rk}\operatorname{Fix}\alpha$ *for an automorphism α of F_n are exactly those allowed by the following inequalities:*

$$\begin{aligned} e + \max(d-1, 0) + \operatorname{rk}\operatorname{Fix}\alpha &\leq n \\ 4e + 2d + 2\operatorname{rk}\operatorname{Fix}\alpha &\leq 3n + 1 \qquad (\leq 3n \text{ if } d = 0). \end{aligned}$$

The first inequality is a strengthening of the bound $\operatorname{rk}\operatorname{Fix}\alpha \leq n$ proved in [4]. If the rank equals n, one must have $e = 0$ and $d \leq 1$, as proved in [5].

Our results have to do with growth. In an appendix, we show:

Theorem 3. *Given $\alpha \in$ Aut (F_n) and an element (or conjugacy class) g in F_n, there exist $\lambda \geq 1$ and $m \in \mathbf{N}$, and $C_1, C_2 > 0$, such that the length of $\alpha^p(g)$ is bounded between $C_1\lambda^p p^m$ and $C_2\lambda^p p^m$ for all $p \geq 1$.*

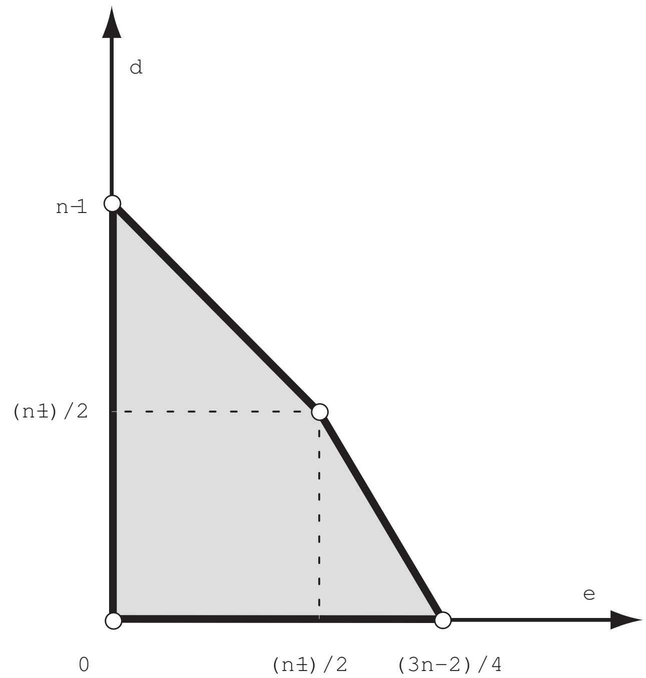

FIGURE 1. Possible values for (e, d).

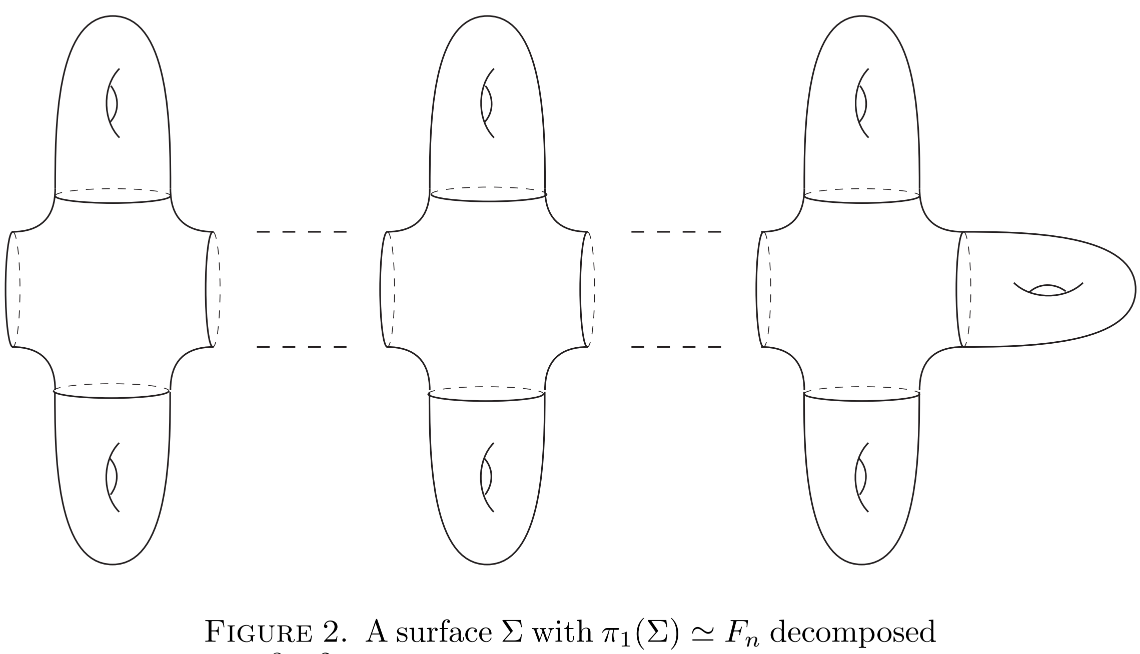

FIGURE 2. A surface Σ with $\pi_1(\Sigma) \simeq F_n$ decomposed into $\frac{3n-2}{4}$ subsurfaces carrying pseudo-Anosov maps.

This result is not deep (if one assumes train tracks), but has never appeared in print. Our proof uses arguments from a preliminary version of [3]. We also explain how to determine the growth types (λ, m) with $\lambda > 1$ from the set of attracting laminations of α and the Perron-Frobenius eigenvalues. As shown in [12], the numbers λ may be viewed as Hölder exponents associated to periodic

points on the boundary of F_n.

Given α, the number e' of growth types (λ, m) with $\lambda > 1$ is bounded by e, and *Theorems 1 and 2 are valid with e replaced by e'.* We also show (Theorem 4.7) that the degree m is bounded by $n/2 - 1$ when $\lambda > 1$ (when $\lambda = 1$, the optimal bound for m is $n - 1$ for growth of conjugacy classes, n for growth of elements of F_n).

Let us now say a few words about the proofs. As mentioned above, one of our results is a strengthening of the "Scott conjecture" $\operatorname{rk} \operatorname{Fix} \alpha \leq n$. As in [9], we work with the outer automorphism Φ determined by α, and instead of $\operatorname{rk} \operatorname{Fix} \alpha$ we use a related invariant r associated to Φ. In the surface case, r equals the absolute value of the Euler characteristic of the whole subsurface Σ_f where h' equals the identity, whereas $\operatorname{rk} \operatorname{Fix} \alpha$ can only see one component of Σ_f.

The proof is by induction on n. If Φ is polynomially growing (i.e. $e = 0$), some power of Φ preserves (up to conjugacy) a nontrivial decomposition of F_n as a free product, or preserves a free factor of rank $n - 1$. This makes induction possible.

If $e > 0$, we consider a Φ-invariant $\mathbf{R}$-tree T with trivial arc stabilizers, as in [9]. Point stabilizers have rank $< n$ and we can argue by induction. The induction starts from the polynomial subgroups of Φ: a canonical finite family of conjugacy classes of subgroups on which all the polynomial growth is concentrated [10]. Our techniques give precise bounds for the ranks of these subgroups (see Theorem 4.1).

Going back to the tree T, the inequality that would make the induction work to prove our main results is $\sum_{i=1}^{b}(3n_i - 2) \leq 3n - 6$, where $G_1, \dots, G_b$ are representatives for conjugacy classes of non-trivial stabilizers of points of T, and $n_i = \operatorname{rk} G_i$. Unfortunately, this inequality is false, as the following example shows.

Suppose that Φ is induced by a homotopy equivalence f of a finite complex Y obtained by attaching the boundary of a once-punctured torus S onto a graph Γ with $\pi_1\Gamma \simeq F_{n-1}$. Assume furthermore that S and Γ are f-invariant, and $f_{|S}$ is pseudo-Anosov. There is a Φ-invariant tree T in which $\pi_1\Gamma$ is a stabilizer, so that $\sum(3n_i - 2) = 3(n-1) - 2 = 3n - 5$ (there is a similar example with S a four-punctured sphere).

To make the induction work, we need to have a more precise control on the outer automorphism Ψ induced on $\pi_1\Gamma$ than stated so far. The key remark is the following. The boundary of S provides a nontrivial conjugacy class $\overline{\gamma}$ which is fixed by Ψ. The element γ is a commutator in F_n, but it cannot be one in $\pi_1\Gamma$. Indeed, it must be a primitive element of $\pi_1\Gamma$, as otherwise $\pi_1 Y$ could not be free. This motivates the introduction of an invariant k, computed from periodic conjugacy classes, so that an inequality $\sum_i(3n_i - 2 - k_i) \leq 3n - 6 - k$ does hold (Proposition 3.1).

In the first, preliminary, section, we explain how the induction works, and we define the polynomial subgroups. In Section 2, we define the basic invariants and we study how they behave under the induction process. In Section 3, we establish the needed inequalities about the numbers n_i. We prove the inequalities of our main theorems in Section 4, as Corollaries 4.5 and 4.6. In Section 5 we construct examples, showing that the inequalities are optimal. This section may

be read independently of the others. As mentioned above, we study growth in an appendix.

Acknowledgement. *This research started in 1996 as joint work with Martin Lustig, and the inequality $e \leq \frac{3n-2}{4}$ is proved in [11]. Though Martin refused to coauthor the present paper, he certainly deserves a lot of credit for it.*

I also thank the referee for a suggestion which simplified the proof of Proposition 3.1 a great deal.

1. Preliminaries

We view $\Phi \in \mathrm{Out}\,(F_n)$ as a collection of automorphisms $\alpha \in \mathrm{Aut}\,(F_n)$. We say that α *represents* Φ, and we write $\Phi = \hat{\alpha}$. We write i_x for the inner automorphism $g \mapsto xgx^{-1}$, so that $\hat{\beta} = \hat{\alpha}$ if and only if $\beta = i_x \circ \alpha$ for some $x \in F_n$. We say that α and β are *isogredient* if $\beta = i_y \circ \alpha \circ (i_y)^{-1}$ for some y (the word "similar" was used in [7] and [9]).

We write $\mathrm{Fix}\,\alpha$ for the fixed subgroup $\mathrm{Fix}\,\alpha = \{g \in F_n \mid \alpha(g) = g\}$, and $\mathrm{rk}\,\mathrm{Fix}\,\alpha$ for its rank. Isogredient automorphisms have conjugate fixed subgroups.

Train tracks and laminations [1].

Given $\Phi \in \mathrm{Out}\,(F_n)$, there exists $q \geq 1$ such that Φ^q is represented by an *improved relative train track map* $f : G \to G$, as in [1, Theorem 5.1.5]. We denote by H_i the *i-th stratum*. The *height* of a path γ is the largest i such that γ contains an edge of H_i. The image of an edge of H_i is a path of height $\leq i$.

There are three types of strata. If e is an edge in a 0-stratum H_i, then $f(e)$ has height $< i$. If H_i is an NEG stratum, it consists of a single edge e_i, and $f(e_i) = e_i u_i$ with u_i of height $< i$. If H_i is exponential, it has a *transition matrix* M whose entry M_{pq} records the number of times that the image of the p-th edge of H_i crosses the q-th edge (in either direction). This matrix is positive, and it has a largest eigenvalue $\lambda > 1$ called the *Perron-Frobenius eigenvalue* of H_i. It is an algebraic integer.

To an exponential stratum H_i is associated an *attracting lamination* Λ_i of Φ [1, Section 3]. It may be defined as the set of bi-infinite paths γ in G such that any finite subpath of γ is contained in some tightened image $f^p_\sharp(e)$, for $p \geq 1$ and e an edge of H_i. These paths γ are the *leaves* of Λ_i. Through the identification between $\pi_1(G)$ and F_n, one may view Λ_i as a lamination on F_n (a subset of the quotient of $\partial F_n \times \partial F_n$ minus the diagonal by the action of $\mathbf{Z}/2\mathbf{Z}$, which interchanges the factors, and the diagonal action of F_n).

We have described the attracting laminations through the choice of a train track representative f, but the set of attracting laminations on F_n depends only on Φ. For any f representing a power of Φ, there is a bijection between the set of attracting laminations of Φ and the set of exponential strata of f.

Growth.

We denote by $\overline{g}$ the *conjugacy class* of $g \in F_n$. The outer automorphism Φ acts on the set of conjugacy classes of elements, and on the set of conjugacy classes of subgroups.

If we fix a free basis for F_n, the *length* $|g|$ is the length of the reduced word representing g. The length $|\overline{g}|$ is the length of any cyclically reduced word representing a conjugate of g.

We say that two sequences a_p and b_p are equivalent, or that a_p *grows like* b_p, if the ratio a_p/b_p is bounded away from 0 and infinity. It is well-known that any sequence $|\alpha^p(g)|$ (resp. $|\Phi^p(\overline{g})|$) grows like a polynomial of degree $m \in \mathbf{N}$, or has exponential growth. We then say that g (resp. $\overline{g}$) grows *polynomially with degree m, or exponentially*, under iteration of α (resp. Φ). This does not depend on the choice of a basis.

If g belongs to a finitely generated α-invariant subgroup H, the growth of g under $\alpha_{|H}$ is the same as its growth under α, because H is quasiconvex in F_n [14]. Similarly, the growth of $\overline{g}$ under $\Phi_{|H}$ is the same as its growth under Φ.

An automorphism Φ (or a representative α) is *polynomially growing* if every conjugacy class grows polynomially. If f represents a power of Φ, this is equivalent to saying that f has no exponential stratum.

These facts are sufficient for the proof of Theorems 1 and 2. In the appendix, we shall prove the more precise result that g and $\overline{g}$ always grow like a sequence $\lambda^p p^m$ with $\lambda \geq 1$ and $m \in \mathbf{N}$. We say that (λ, m) is the *growth type*. If $\lambda = 1$, the growth is polynomial. If $\lambda > 1$, the growth is exponential and we say that λ is the *exponential growth rate*. If Φ^q is represented by an improved train track map, then any exponential growth rate is the q-th root of the Perron-Frobenius eigenvalue of an exponential stratum.

Setting up the induction.

We shall prove our main results by induction on n. First we replace $\Phi \in \mathrm{Out}\,(F_n)$ by a power, so that it is represented by an improved relative train track map $f : G \to G$.

First suppose that Φ is polynomially growing. We consider the highest stratum. It consists of a single edge E.

Definition 1.1. *Let Φ be polynomially growing, represented by $f : G \to G$.*

(1) *If there is a non-trivial decomposition $F_n = G_1 * G_2$ such that Φ has a representative α which satisfies $\alpha(G_i) = G_i$, we let n_i be the rank of G_i and we denote $\alpha_i = \alpha_{|G_i}$, so that $\alpha = \alpha_1 * \alpha_2$. This happens in particular whenever the top edge E separates G.*

(2) *If E does not separate, there is a decomposition $F_n = G_1 * \langle t \rangle$ (with G_1 of rank $n-1$) and a representative α of Φ such that $\alpha(G_1) = G_1$ and $\alpha(t) = tu$ with $u \in G_1$. We assume that u cannot be written $u = a\alpha(a^{-1})$ with $a \in G_1$, since otherwise $\alpha(ta) = ta$ and we reduce to the previous case $\alpha = \alpha_1 * \alpha_2$. We denote $\alpha_1 = \alpha_{|G_1}$.*

This will allow us to deduce properties of α from properties of the α_i's.

In the remainder of this subsection, we assume that Φ is not polynomially growing. We then use the Φ-invariant $\mathbf{R}$-tree T associated to the highest exponential stratum H, in the following sense.

If H is the highest of all strata, T is the tree constructed in section 2 of [7]. If not, we adapt the construction of [7] as follows. Let H' be the union of H and all (non-exponential) strata above it. We consider the transition matrix $M_{H'}$: there is one row and one column for each edge of H', and each entry records the number of times that the image of an edge crosses an edge (in either direction). The eigenvalues of $M_{H'}$ are those of the transition matrix M_H of H, together with 0's and 1's. In particular, $M_{H'}$ has a non-negative eigenvector associated to the Perron-Frobenius eigenvalue $\lambda > 1$ of M_H. We then apply the construction of [7], using this eigenvector to give a PF-length to edges of H' (edges below H have PF-length 0).

In either case, the group F_n acts on the $\mathbf{R}$-tree T isometrically, minimally, with no global fixed point. All arc stabilizers are trivial.

We let $\ell : F_n \to [0, \infty)$ be the length function of T, defined by $\ell(g) = \min_{x \in T} d(x, gx)$. Its value on g only depends on the conjugacy class $\overline{g}$, so we sometimes write $\ell(\overline{g})$. An element g, or its class $\overline{g}$, is elliptic if $\ell(g) = 0$, hyperbolic if $\ell(g) > 0$. An elliptic element has a unique fixed point, a hyperbolic element has an axis along which it acts as translation by $\ell(g)$. Any $g \in F_n$ represented by a loop meeting only strata below H is elliptic.

The tree T is Φ-invariant, in the sense that $\ell(\Phi(\overline{g})) = \lambda \ell(\overline{g})$, with $\lambda > 1$ the Perron-Frobenius eigenvalue of M_H as above.

As in [12, §3], we note:

Lemma 1.2. *If $\overline{g}$ grows polynomially, then g is elliptic in T (it fixes a unique point).*

Proof. $\lambda^p \ell(\overline{g}) = \ell(\Phi^p(\overline{g}))$ is bounded by a constant times the word length of $\Phi^p(\overline{g})$, so $\ell(\overline{g}) = 0$. $\square$

Stabilizers of points of T have rank $< n$, and there are finitely many orbits of points with non-trivial stabilizer. These are general facts about trees with trivial arc stabilizers [8]. In the case at hand, they may be deduced from the description of stabilizers used in Section 3.

Definition 1.3. *If Φ is exponentially growing, let T be the Φ-invariant $\mathbf{R}$-tree constructed above. We let m_i, $1 \leq i \leq b$, be representatives for the orbits of points with non-trivial stabilizer. We denote by G_i the stabilizer of m_i (it is malnormal, but not always a free factor). It has rank $n_i < n$. The conjugacy classes of the G_i's are permuted by Φ. Replacing Φ by a power, we may assume that Φ leaves G_i invariant (up to conjugacy) and therefore induces $\Phi_i \in \mathrm{Out}\,(G_i)$.*

Polynomial subgroups.

Let $\Phi \in \mathrm{Out}\,(F_n)$. A subgroup P is *polynomial* if there exist $q \geq 1$ and α representing Φ^q such that $\alpha(P) = P$ and $\alpha_{|P}$ is polynomially growing. If P is polynomial, so are its conjugates and its images by any automorphism representing a power of Φ. Furthermore, Φ and all its powers have the same polynomial subgroups (for negative powers, recall that the inverse of a polynomially growing automorphism is polynomially growing).

Proposition 1.4. *Let $\Phi \in \mathrm{Out}(F_n)$.*

(1) *The conjugacy class $\overline{g}$ of $g \in F_n$ grows polynomially if and only if g belongs to a polynomial subgroup.*
(2) *Every non-trivial polynomial subgroup is contained in a unique maximal one.*
(3) *If $g \neq 1$ grows polynomially under α representing Φ, the maximal polynomial subgroup $P(g)$ containing g is the set of elements growing polynomially under α.*
(4) *Maximal polynomial subgroups have finite rank, are malnormal, and there are finitely many conjugacy classes of them.*

Inequalities for the ranks of the polynomial subgroups will be given in Theorem 4.1. For the automorphism θ_n constructed in Section 5, the group P_0 is a maximal polynomial subgroup.

Proof. By induction on n. The results are true if Φ is polynomially growing, with F_n the unique maximal polynomial subgroup. If not, we consider T, G_i, Φ_i as in 1.3 (after replacing Φ by a power if needed). Any polynomial subgroup of G_i (relative to Φ_i) is a polynomial subgroup of F_n (relative to Φ).

Consider $g \neq 1$ such that $\overline{g}$ grows polynomially (this is the case, in particular, if g belongs to a polynomial subgroup). Let $P(g)$ be the set of $h \in F_n$ such that both $\overline{h}$ and $\overline{gh}$ grow polynomially. We prove by induction on n that $P(g)$ is a polynomial subgroup.

Since $\overline{g}$ grows polynomially, the element g fixes a point in T by Lemma 1.2, and by conjugating g we may assume $g \in G_i$. If $h \in P(g)$, then h is also elliptic. It fixes the same point as g because otherwise gh would be hyperbolic in T, contradicting polynomial growth of $\overline{gh}$. This shows $P(g) \subset G_i$. By induction, $P(g)$ is a polynomial subgroup (relative to Φ_i, hence also to Φ).

Clearly $P(g)$ contains g, as well as every polynomial subgroup containing g: it is the largest polynomial subgroup containing g. This shows assertions (1) and (2).

Suppose g and α are as in (3). As above, we may assume $g \in G_i$. Any element growing polynomially under α is in $P(g)$. For the converse, note that $\alpha(g)$ belongs to $P(g)$, hence to G_i. We deduce that G_i and $\alpha(G_i)$ have a non-trivial intersection, hence are equal (they are point stabilizers in a tree with trivial arc stabilizers). By induction, any $h \in P(g)$ grows polynomially under α.

The finiteness statements in (4) are immediate by induction. Let us prove malnormality. Let P be a maximal polynomial subgroup. If P and gPg^{-1} meet non-trivially, they are equal (by the uniqueness statement in (2)), so we only have to show that P equals its normalizer. As above, we may assume $P \subset G_i$. Since P fixes a unique point m_i in T, any g normalizing P belongs to G_i. By induction, P equals its normalizer in G_i, so it also equals its normalizer in F_n. $\square$

Lemma 1.5. *Given $\Phi \in \mathrm{Out}(F_n)$, the following are equivalent:*

(1) *There exists a polynomial subgroup of rank ≥ 2.*

(2) *There exist* $q \geq 1$ *and* α *representing* Φ^q *with* $\operatorname{rk} \operatorname{Fix} \alpha \geq 2$.

Proof. (2) obviously implies (1). For the converse, it suffices to prove (2) under the assumption that Φ is polynomially growing and $n \geq 2$. We distinguish two cases, as in Definition 1.1. First suppose that some power of Φ has a representative $\alpha = \alpha_1 * \alpha_2$. If n_1 or n_2 is ≥ 2, we use induction. Otherwise, α^2 is the identity of F_2 and (2) holds. In the second case, we consider a decomposition $F_n = G_1 * \langle t \rangle$. If $n \geq 3$, we use induction. If $n = 2$, then $\alpha^2 \in \operatorname{Aut}(F_2)$ is of the form $a \mapsto a, t \mapsto ta^i$ for some $i \in \mathbf{Z}$, and its fixed subgroup has rank 2 (if $i \neq 0$, it is generated by a and tat^{-1}). □

Corollary 1.6. *Let* $\Phi \in \operatorname{Out}(F_n)$. *Suppose that some non-trivial conjugacy class* $\overline{g}$ *grows polynomially. If* $\overline{g}$ *is not periodic, there exist* $q \geq 1$ *and* α *representing* Φ^q *with* $\operatorname{rk} \operatorname{Fix} \alpha \geq 2$. *If* $\overline{g}$ *is periodic, there exists* α *with* $\operatorname{rk} \operatorname{Fix} \alpha \geq 1$.

Proof. Only the first assertion requires a proof. By Proposition 1.4, g belongs to a polynomial subgroup. It has rank at least 2, as otherwise $\overline{g}$ would be periodic. We conclude by Lemma 1.5. □

We also record the following easy fact:

Lemma 1.7. *If every conjugacy class is* Φ*-periodic, then* Φ *has finite order.* □

In fact Φ has finite order as soon as every conjugacy class of length ≤ 2 is periodic.

2. The basic invariants

We write $x^+ = \max(x, 0)$.

Given $\Phi \in \operatorname{Out}(F_n)$, we now define numbers e, s, d, p, r, k. If we consider several automorphisms simultaneously, we write $e(\Phi), s(\Phi)$,... so that no confusion arises. If we consider $\alpha \in \operatorname{Aut}(F_n)$, we write $e, s, \ldots$ for the invariants of the outer automorphism $\Phi = \hat{\alpha}$ represented by α.

- e is the number of exponential strata of any improved relative train track map representing a power of Φ. It is also the number of attracting laminations of Φ [1, Subsection 3.1]. Note that e is an upper bound for the number of growth types (λ, m) with $\lambda > 1$ (see Theorem 6.2), and that $e = 0$ if and only if Φ is polynomially growing. Also note that $e(\Phi^{-1}) = e(\Phi)$ by Subsection 3.2 of [1].
- s is the maximal length of a chain of attracting laminations $\Lambda_0 \supsetneqq \cdots \supsetneqq \Lambda_s$. Any growth type (λ, m) with $\lambda > 1$ satisfies $m \leq s$ (see appendix).
- d is the maximal degree of polynomial growth of conjugacy classes. If $e = d = 0$, then Φ has finite order by Lemma 1.7. One has $d(\Phi^{-1}) = d(\Phi)$ by [13] (to be precise, one also needs Proposition 1.4 to reduce to the polynomially growing case studied in [13], and Lemma 2.3 below to compare d to growth of elements).
- $p = (d-1)^+ = \max(d-1, 0)$.
- r is the index of Φ, computed using ranks of fixed subgroups. Namely, we write $r_0(\alpha) = (\operatorname{rk} \operatorname{Fix} \alpha - 1)^+ = \max(\operatorname{rk} \operatorname{Fix} \alpha - 1, 0)$ for α representing Φ. We then define $r = \sum_i r_0(\alpha_i)$, the sum being taken over a set of representatives of

isogredience classes (recall that α, α' are isogredient if there is an inner automorphism conjugating them). One has $r \leq n-1$ by [4, Corollary 6.4] (see also [9]). If $\operatorname{rk}\operatorname{Fix}\alpha \geq 2$, we say that the isogredience class of α contributes $r_0(\alpha)$ to r. Only finitely many classes contribute.

• k is the rank of the subgroup generated by Φ-periodic conjugacy classes in the abelianization of F_n (so $k \leq n$). This number plays an essential role in the proof of our main results (see the discussion in the introduction and in Section 3).

The numbers e, s, d, p, k do not change if Φ is replaced by a positive power (and e, d, p, r, k do not change if Φ is replaced by Φ^{-1}; we do not know whether s is the same for Φ and Φ^{-1}). The following lemma controls r, so that we can always replace Φ by a power when proving upper bounds for the invariants.

Lemma 2.1. *$r(\Phi) \leq r(\Phi^q)$ for any $q \geq 1$.*

Proof. We have to bound any finite sum $\sum_i r_0(\alpha_i)$ as above by $r(\Phi^q)$. One always has $\operatorname{rk}\operatorname{Fix}\alpha \leq \operatorname{rk}\operatorname{Fix}\alpha^q$, since $\operatorname{Fix}\alpha$ is a free factor of $\operatorname{Fix}\alpha^q$ by [6], so $r_0(\alpha_i) \leq r_0(\alpha_i^q)$. But this is not enough, because α_i^q and α_j^q may be isogredient for $i \neq j$ even though α_i, α_j aren't.

We may assume that α_i^q and α_j^q are equal if they are isogredient (by changing automorphisms within their isogredience class), so we reduce to showing

$$\sum_{\alpha_i^q = \beta} r_0(\alpha_i) \leq r_0(\beta)$$

for any β representing Φ^q. We may also assume that $\operatorname{Fix}\beta$ has rank ≥ 2.

Note that $\operatorname{Fix}\beta$ is α_i-invariant and contains $\operatorname{Fix}\alpha_i$. We claim that the restrictions of the α_i's to $\operatorname{Fix}\beta$ represent the same outer automorphism $\tilde{\Phi}$ (of finite order), and define distinct isogredience classes. Assuming this, we conclude because

$$\sum_{\alpha_i^q = \beta} r_0(\alpha_i) \leq r(\tilde{\Phi}) \leq \operatorname{rk}\operatorname{Fix}\beta - 1 = r_0(\beta).$$

To prove the claim, fix distinct i, j. We write $\alpha_i = i_h \circ \alpha_j$ (with $i_h(g) = hgh^{-1}$) and we show $h \in \operatorname{Fix}\beta$. We have $\alpha_i^q = i_n \circ \alpha_j^q$ with $n = h\alpha_j(h)\dots\alpha_j^{q-1}(h)$. We deduce $n = 1$, and this implies $\alpha_j^q(h) = h$, so $h \in \operatorname{Fix}\beta$ as required. This shows the first assertion. Next (as in [9, 2.4]), suppose that i_m conjugates the restrictions of α_i and α_j, with $m \in \operatorname{Fix}\beta$. Then $\alpha_i\alpha_j^{-1} = i_{m\alpha_j(m^{-1})}$ on $\operatorname{Fix}\beta$, hence everywhere because $\alpha_i\alpha_j^{-1}$ is inner and $\operatorname{Fix}\beta$ has rank ≥ 2. This means that i_m conjugates α_i and α_j on the whole of F_n, so α_i, α_j are isogredient, a contradiction to $i \neq j$. ☐

Remark 2.2. By Corollary 1.6, $d(\Phi) > 0$ implies that there exists $q \geq 1$ with $r(\Phi^q) > 0$.

The following lemma compares growth of conjugacy classes and growth of elements.

Lemma 2.3. *Let $\Phi \in \mathrm{Out}\,(F_n)$, and $d = d(\Phi)$. Suppose that some $g \in F_n$ grows polynomially with degree $d' > d$ under iteration of a representative α of Φ. Then:*

(1) *$d' = d + 1$.*
(2) *If $\mathrm{Fix}\,\alpha$ is non-trivial, then $d' = 1$ (and therefore $d = 0$).*

Examples. If α is a non-trivial inner automorphism, then $d = 0$ and any non-trivial g grows linearly ($d' = 1$). In the example $a \mapsto bab^{-1}$, $b \mapsto b^2ab^{-1}$, due to Bridson-Groves, $\overline{b}$ grows linearly and b grows quadratically (the fixed subgroup is trivial).

Proof. We first prove (2). Fix a basis of F_n, and $h \neq 1$ in $\mathrm{Fix}\,\alpha$. Write $\alpha^p(g) = w_p g_p w_p^{-1}$ with g_p cyclically reduced. The length of w_p grows with degree d', because g_p grows with degree at most d. We claim that the cancellation between the initial segments of w_p and hw_p grows at most linearly. Assuming this for the moment, we consider $\alpha^p(hg) = hw_p g_p w_p^{-1}$. If $d' > 1$, the cyclic reduction of this word grows with degree d', so $\overline{hg}$ grows with degree d' under iteration of Φ. This contradicts $d' > d$.

To prove the claim, first note that the cancellation between $\alpha^p(g)$ and $h^\infty = \lim_{p\to\infty} h^p$ grows at most linearly (see [12, p. 424]). The same holds for the cancellation between w_p and h^∞, and therefore for that between w_p and hw_p.

For (1), we let α_1 be the restriction of α to the polynomial subgroup $P(g)$ (see item (3) in Proposition 1.4). Let Φ_1 be the outer automorphism determined by α_1. If $d(\Phi_1) = 0$, then Φ_1 has finite order by Lemma 1.7, and g grows at most linearly. If not, we use Corollary 1.6. After replacing α by a power, we write $\alpha_1 = i_a \circ \beta$, where $a \in P(g)$ and $\beta \in \mathrm{Aut}\,(P(g))$ has a non-trivial fixed subgroup. Since $d(\Phi_1) > 0$, assertion (2) proved above implies that a and g grow with degree at most d under iteration of β. Then

$$\alpha^n(g) = [a\beta(a)\beta^2(a)\dots\beta^{n-1}(a)]\ \beta^n(g)\ [a\beta(a)\beta^2(a)\dots\beta^{n-1}(a)]^{-1}$$

grows with degree $\leq d + 1$. □

Our next goal will be to understand the behavior of the invariants in the induction process, so we consider Φ as in 1.1 or 1.3.

First suppose $\alpha = \alpha_1 * \alpha_2$ as in the first item of Definition 1.1. We denote by k_i, r_i, d_i the invariants associated to $\hat{\alpha}_i$.

Lemma 2.4.

(1) *If $\mathrm{Fix}\,\alpha_1$ or $\mathrm{Fix}\,\alpha_2$ is trivial, then $r = r_1 + r_2$ and $\max(d_1, d_2) \leq d \leq 1 + \max(d_1, d_2)$.*
(2) *If $\mathrm{Fix}\,\alpha_1$ and $\mathrm{Fix}\,\alpha_2$ are both non-trivial, then $r = r_1 + r_2 + 1$ and d equals 1 or $\max(d_1, d_2)$.*
(3) *$k = k_1 + k_2$.*

Proof. The assertions about r are proved in [9]. Here is the idea. If an isogredience class other than that of α contributes to r, it contributes the same amount to

either r_1 or r_2 (but not both). The class of α contributes $(\operatorname{rk}\operatorname{Fix}\alpha_i - 1)^+$ to r_i and $(\operatorname{rk}\operatorname{Fix}\alpha - 1)^+$ to r. To conclude, note that $\operatorname{Fix}\alpha = \operatorname{Fix}\alpha_1 * \operatorname{Fix}\alpha_2$.

The number d is bounded from below by $\max(d_1, d_2)$, and from above by the maximal degree of growth of elements of G_i under α_i, so the assertions about d follow from Lemma 2.3. The assertion about k is easy and left to the reader. □

We now consider a decomposition $F_n = G_1 * \langle t\rangle$ and a representative α of Φ such that $\alpha(G_1) = G_1$ and $\alpha(t) = tu$ with $u \in G_1$, as in item (2) of Definition 1.1. Let k_1, r_1, d_1 be the invariants associated to the outer automorphism Φ_1 of G_1 represented by $\alpha_1 = \alpha_{|G_1}$.

Lemma 2.5.

(1) $r \leq r_1 + 1$, *with equality if and only if both* α_1 *and* $i_u \circ \alpha_1$ *have non-trivial fixed subgroups.*
(2) $d \leq d_1 + 1$.
(3) $k \leq k_1 + 1$, *with equality if and only there exists* $a \in G_1$ *such that* ta *is* α*-periodic.*

Proof. The assertion about r is proved in [9]. One now has $\operatorname{Fix}\alpha = \operatorname{Fix}\alpha_1 * t\operatorname{Fix}(i_u \circ \alpha_1)t^{-1}$. The contribution of the isogredience class of α to r is $(\operatorname{rk}\operatorname{Fix}\alpha_1 + \operatorname{rk}\operatorname{Fix}(i_u \circ \alpha_1) - 1)^+$, whereas its contribution to r_1 is $(\operatorname{rk}\operatorname{Fix}\alpha_1 - 1)^+ + (\operatorname{rk}\operatorname{Fix}(i_u \circ \alpha_1) - 1)^+$.

Let us prove (2). As in the proof of Lemma 2.3, replace α by a power and write $\alpha_1 = i_a \circ \beta_1$, where $a \in G_1$ and $\operatorname{Fix}\beta_1$ is non-trivial, using Corollary 1.6; if $d_1 = 0$, we take β_1 to be the identity, using Lemma 1.7. By Lemma 2.3, all elements of G_1 grow with degree at most d_1 under β_1. Now $\beta = (i_a)^{-1} \circ \alpha$ sends t to an element of the form $u'tu''$, so t grows with degree at most $d_1 + 1$ under β. This implies $d \leq d_1 + 1$.

We now study k. Represent conjugacy classes of F_n as cyclic words whose letters are either $t^{\pm 1}$ or non-trivial elements of G_1. Let $\overline{w}$ be a cyclic word not contained in G_1. Split it before every letter t and after every t^{-1}. This expresses $\overline{w}$ as a product of subwords of the form tat^{-1}, ta, at^{-1}, a, with $a \in G_1$ and a non-trivial in subwords tat^{-1} or a. This decomposition of $\overline{w}$ is preserved by α, so that $\overline{w}$ is Φ-periodic if and only if every subword is an α-periodic element.

Let K be the quotient of the abelianization of F_n by the subgroup generated by all Φ_1-periodic conjugacy classes (of elements of G_1). It has torsion-free rank $n - k_1$. To bound k, we need to control the image in K of α-periodic elements of the form tat^{-1} or ta (we treat at^{-1} as the inverse of ta^{-1}).

If tat^{-1} is α-periodic, then a is $(i_u^{-1} \circ \alpha_1)$-periodic, so $\overline{a}$ is Φ_1-periodic and tat^{-1} maps trivially to K. On the other hand, the image of ta in K has infinite order. But, if ta and ta' are both α-periodic, then $a^{-1}a'$ is α_1-periodic and therefore ta, ta' have the same image in K. This proves the lemma. □

We now suppose that Φ has exponential growth, and we use the $\mathbf{R}$-tree T as in 1.3. Recall that we have defined invariant subgroups G_i and restrictions Φ_i. The

following lemma allows us to bound the invariants of Φ in terms of those of the Φ_i's (which we denote with the subscript i). All sums are over i.

Lemma 2.6. *The invariants of Φ satisfy:*

$$\begin{aligned} e &\leq 1 + \sum e_i \\ s &\leq 1 + \max s_i \\ d &= \max d_i \\ r &= \sum r_i \\ k &\leq \sum k_i. \end{aligned}$$

Proof. The results for d and k are direct consequences of Lemma 1.2 (note that for $g \in G_i$ the growth of $\overline{g}$ is the same for Φ_i as for Φ). The equality for r is proved in [9]: if an isogredience class contributes to r, it contributes the same amount to exactly one r_i.

As in the previous section, let H be the highest exponential stratum of the train track, and let H' be the union of H with all the strata above it. Every exponential stratum $\tilde{H}$ other than H is contained in a component $\tilde{G}$ of the closure of $G \setminus H'$. It follows from the way T was constructed (see [7] and the previous section) that the fundamental group of $\tilde{G}$ fixes a point of T, whose stabilizer is conjugate to some G_i. There is a bijection between exponential strata and attracting laminations [1, Definition 3.1.12]. By [1, Definition 3.1.5], the lamination associated to $\tilde{H}$ is an attracting lamination of Φ_i. This shows the bound for e. If $\Lambda_0 \supsetneq \cdots \supsetneq \Lambda_s$ is a chain of laminations, then $\Lambda_1, \ldots, \Lambda_s$ are attracting laminations of some Φ_i, so $s - 1 \leq s_i$. □

3. Bounding ranks of stabilizers

In this section we assume that Φ is not polynomially growing, and we consider the invariant $\mathbf{R}$-tree T as in 1.3.

Proposition 3.1. *Let n_i be the rank of the stabilizer G_i, and $k_i = k(\Phi_i)$. Then:*

$$\sum_{i=1}^{b} (n_i - 1) \leq n - 2 \tag{1}$$

$$\sum_{i=1}^{b} (3n_i - 2 - k_i) \leq 3n - 6 - k. \tag{2}$$

If equality holds in (1), there is only one exponential stratum.

Recall that $k \leq \sum k_i$, so (2) is implied by the simpler inequality $\sum_i (3n_i - 2) \leq 3n - 6$. But this simpler inequality is not always true, for instance if Φ is induced

by a pseudo-Anosov homeomorphism of a punctured torus or a four-punctured sphere. This explains the introduction of the invariant k.

Proof of Proposition 3.1.

Recall that T was constructed using the highest exponential stratum H of $f : G \to G$. A description of the G_i's is provided by Theorem 6.0.1 of [1], in terms of a subgraph $Z \subset G$.

• First suppose that H is the highest stratum. Then Z is the union of all strata below H.

If H is not geometric (in the sense of Definition 5.1.4 of [1]), then the G_i's are (up to conjugacy) the fundamental groups of the non-contractible components of Z. In particular they are free factors and $\sum n_i \leq n$. If $b = 0$, we have $n \geq 3$ since every automorphism of F_2 is induced by a homeomorphism of a punctured torus. If $b = 1$ or $b = 2$, the existence of the exponential stratum H prevents n_1, n_2 from being too big. More precisely, we have $n_1 < n - 1$ if $b = 1$ by [1, Lemma 3.2.1], and similarly $n_1 + n_2 < n$ if $b = 2$.

We get $\sum(n_i - 1) \leq n - 3$ and $\sum(3n_i - 2) \leq 3n - 6$, with equality in the second inequality possible only if $b = 0$ or $b = 3$. The proposition is true in this case since $\sum k_i \leq k$.

If H is geometric, the components of Z account for all G_i's but one. The exceptional one, say G_b, is cyclic, generated by the homotopy class of a loop ρ (an indivisible Nielsen path) based at a point $v \notin Z$. If b is different from 1 and 4, the previous argument yields $\sum_{i<b}(n_i - 1) \leq n - 3$ and $\sum_{i<b}(3n_i - 2) \leq 3n - 7$, and the proposition holds since $n_b = 1$.

If $b = 1$, we have $\sum(n_i - 1) = 0 \leq n - 2$. On the other hand, $\sum(3n_i - 2) = 1$. It is bounded by $3n - 6$, except if $n = 2$. But in this case Φ is induced by a homeomorphism of a punctured torus, so $k = 0$ (whereas $k_1 = 1$). We get $3n_1 - 2 - k_1 = 0 = 3n - 6 - k$.

When $b = 4$, we have $\sum(n_i - 1) \leq n - 3$ and $\sum_{i\leq 4}(3n_i - 2) \leq 3\sum_{i<4} n_i - 5$. If $\sum_{i<4} n_i < n$, we are done. If not, we prove (2) by showing $k < \sum_{i\leq 4} k_i$.

The geometric stratum H is associated to a connected surface S, as in Definition 5.1.4 of [1]. This surface has four boundary components C_i, with C_4 identified to ρ. It is a punctured sphere because $\sum_{i<4} n_i = n$. The C_i's represent Φ-periodic conjugacy classes z_i, and the relation $\sum z_i = 0$ holds in the abelianization of F_n. Since z_4 may be expressed in terms of z_1, z_2, z_3, we get $k \leq \sum_{i<4} k_i = (\sum_{i\leq 4} k_i) - 1$.

Note that $\sum(n_i - 1) = n - 2$ is possible only if $n = 2$ (and then the exponential stratum is unique).

• Now suppose that there are non-exponential strata above H. Then Z contains all strata below H, no edge of H, and possibly edges from strata above H.

Consider the union of H and all strata below it. By aperiodicity, this subgraph has a component containing H, we call it Y^0. Then define connected subgraphs $Y^0 \subset Y^1 \subset \cdots \subset Y^q = G$ such that $Y^j \setminus Y^{j-1}$ contains exactly one edge.

If H is not geometric, the n_i's are the Betti numbers of the non-contractible components of Z. Define $Z^j = Z \cap Y^j$. Let n_i^j be the Betti numbers of the

non-contractible components of Z^j, and n^j the Betti number of Y^j. Since Y^0 is f-invariant, the previous argument yields $\sum(n_i^0-1) \leq n^0-3$ and $\sum(3n_i^0-2) \leq 3n^0-6$. Using induction on j, one shows $\sum(n_i^q-1) \leq n^q-3$ and $\sum(3n_i^q-2) \leq 3n^q-6$ (when passing from j to $j+1$, the left hand sides cannot increase more than the right hand sides). This proves the proposition since $Z^q = Z$.

If H is geometric, we again have to consider ρ and v. We define $Z^j = (Z \cap Y^j) \cup \{v\}$. The point v is an isolated point of Z^0. The numbers n_i^j and n^j are defined as before, except that we add 1 to the Betti number of the component Z_v^j of Z^j which contains v (in particular, we always consider it as non-contractible). The ranks n_i of the groups G_i are the n_i^q's (the exceptional group G_b is generated by $\pi_1(Z_v^q)$ and the class of ρ).

The inequalities $\sum(n_i^j-1) \leq n^j-2$ and $\sum(3n_i^j-2) \leq 3n^j-5$ are true for $j=0$, hence for $j=q$ by induction. If $\sum(3n_i^j-2) = 3n^j-5$ holds for $j=q$, it holds for $j=0$ so there is a punctured sphere S as above. Let z be the conjugacy class represented by ρ. Removing z from the set of periodic conjugacy classes does not change the subgroup generated in the abelianization of F_n. In other words, z does not contribute to k. But it contributes to k_b since it generates a free factor in G_b. We get $k < \sum k_i$, and the proposition is proved. □

4. Proof of the main results

In this section we give upper bounds for the invariants e, s, d, p, r introduced in Section 2. Recall that we are free to replace Φ by a power (see Lemma 2.1 and the paragraph preceding it).

Theorem 4.1. *Given $\Phi \in \mathrm{Out}\,(F_n)$, let P_j be representatives for the conjugacy classes of maximal polynomial subgroups. Then*

$$e + \sum_j (\mathrm{rk}\, P_j - 1)^+ \leq n-1$$

$$4e + k + 2\sum_j (\mathrm{rk}\, P_j - 1)^+ \leq 3n-2.$$

Recall that $x^+ = \max(x, 0)$.

Proof. The result is trivially true if Φ is polynomially growing (with $e=0$ and $k \leq n$), so we replace Φ by a power and we consider T, G_i, n_i, Φ_i as in 1.3.

If the action on T is free, then by Lemma 1.2 there is no non-trivial polynomially growing conjugacy class (so $k=0$) and no non-trivial polynomial subgroup. Furthermore, $e=1$. The theorem is true in this case since $n \geq 2$, so we assume $b > 0$. We argue by induction on n.

We may assume that each P_j is contained in a (unique) G_{i_j}. The groups P_j contained in a given G_i are non-conjugate maximal polynomial subgroups of Φ_i.

Successively using Lemma 2.6, the induction hypothesis, and Proposition 3.1, we now write:

$$\begin{aligned}
e + \sum_j (\operatorname{rk} P_j - 1)^+ &\leq 1 + \sum_{i=1}^b e_i + \sum_{i=1}^b \sum_{i_j = i} (\operatorname{rk} P_j - 1)^+ \\
&\leq 1 + \sum_{i=1}^b (n_i - 1) \\
&\leq n - 1
\end{aligned}$$

and

$$\begin{aligned}
4e + k + 2\sum_j (\operatorname{rk} P_j - 1)^+ &\leq 4 + 4\sum_{i=1}^b e_i + k + 2\sum_{i=1}^b \sum_{i_j = i} (\operatorname{rk} P_j - 1)^+ \\
&\leq 4 + k + \sum_{i=1}^b (3n_i - 2 - k_i) \\
&\leq 4 + 3n - 6 \\
&\leq 3n - 2.
\end{aligned}$$

□

Proposition 4.2. *If $\Phi \in \operatorname{Out}(F_n)$ is polynomially growing, then $p + r \leq n - 1$.*

Proof. By induction on n. After replacing Φ by a power, we may assume that we are in the situation of Definition 1.1. There are two possibilities.

• First suppose that some α representing Φ preserves a non-trivial decomposition $F_n = G_1 * G_2$. Let n_i be the rank of G_i. Define $\alpha_i, \Phi_i, d_i, p_i, r_i$ by considering the automorphism induced on G_i. Recall from Lemma 2.4 that $r = r_1 + r_2 + 1$ if both $\operatorname{Fix} \alpha_1$ and $\operatorname{Fix} \alpha_2$ are non-trivial, $r = r_1 + r_2$ otherwise. Furthermore, $d \leq \max(d_1, d_2) + 1$ and $p \leq \max(p_1, p_2) + 1$.

If $r = r_1 + r_2$, we write

$$p + r \leq p_1 + p_2 + 1 + r_1 + r_2 \leq n_1 - 1 + n_2 - 1 + 1 = n - 1,$$

so we assume $r = r_1 + r_2 + 1$. By Lemma 2.4, we have $d = 1$ or $d = \max(d_1, d_2)$. In both cases $p = \max(p_1, p_2)$ and

$$p + r \leq p_1 + p_2 + r_1 + r_2 + 1 \leq n - 1.$$

• Now suppose that there is a decomposition $F_n = G_1 * \langle t \rangle$ and a representative α of Φ which leaves $G_1 \simeq F_{n-1}$ invariant and maps t to tu with $u \in G_1$. As in 1.1, we assume that u cannot be written as $a\alpha(a^{-1})$ with $a \in G_1$. In particular, $u \neq 1$. Define $\alpha_1, \Phi_1, d_1, p_1, r_1$ by restricting to G_1. Recall (Lemma 2.5) that $r \leq r_1 + 1$, with equality if and only if both α_1 and $i_u \circ \alpha_1$ have non-trivial fixed subgroups, and $p \leq p_1 + 1$.

We show that $r = r_1 + 1$ implies $p = p_1$, so that in all cases

$$p + r \leq p_1 + r_1 + 1 \leq n_1 - 1 + 1 = n - 1.$$

Suppose $r = r_1 + 1$. Since $\mathrm{Fix}\,(i_u \circ \alpha_1)$ is non-trivial, there is a non-trivial α-fixed element of the form txt^{-1}, and u is α-fixed by property ne-(iii) of [1, Theorem 5.1.5]. In particular, the element t grows linearly under α. If $d \geq 2$, some element or conjugacy class of G_1 grows with degree d, and $d = d_1$ by Lemma 2.3. If $d \leq 1$, we have $p = p_1$. $\square$

Combining Theorem 4.1 and Proposition 4.2, we deduce:

Theorem 4.3. *Given any $\Phi \in \mathrm{Out}\,(F_n)$, we have:*

$$\begin{gathered} e + p + r \leq n - 1 \\ 4e + 2p + 2r + k \leq 3n - 2. \end{gathered}$$

Proof. By Theorem 4.1, it suffices to show $p + r \leq \sum_j (\mathrm{rk}\, P_j - 1)^+$. We may replace Φ by a power, and so assume that each P_j is fixed (up to conjugacy). Since P_j equals its normalizer, there is a well-defined induced Φ_j, with associated p_j and r_j, and $p = \max p_j$.

Let α be a representative of Φ. If $\mathrm{Fix}\,\alpha$ is non-trivial, it is contained in a conjugate of some P_j, and by changing α in its isogredience class we may assume $\mathrm{Fix}\,\alpha \subset P_j$. This implies that P_j is α-invariant (by uniqueness of maximal polynomial subgroups). Furthermore, if $\mathrm{Fix}\,\alpha$ and $\mathrm{Fix}\,\alpha'$ are contained in P_j, then the restrictions of α and α' to P_j represent the same outer automorphism (if $\alpha' = i_h \circ \alpha$, then P_j is i_h-invariant, so $h \in P_j$ because P_j equals its normalizer); if P_j has rank ≥ 2, and α, α' are not isogredient, the restrictions are not isogredient (see [9] or the proof of Lemma 2.1).

Write $r = \sum r_0(\alpha_m)$, the sum being taken over non-isogredient automorphisms α_m. We may assume $\mathrm{Fix}\,\alpha_m \subset P_{j_m}$, with P_{j_m} of rank ≥ 2. Now

$$p + r \leq \sum_j p_j + \sum_j \sum_{\mathrm{Fix}\,\alpha_m \subset P_j} r_0(\alpha_m) \leq \sum_j p_j + \sum_j r_j \leq \sum (\mathrm{rk}\, P_j - 1)^+$$

by Proposition 4.2. $\square$

The inequality $4e + 2p + 2r + k \leq 3n - 2$ may be an equality, for instance for the automorphism α of F_3 defined by $a \mapsto a$, $b \mapsto b$, $c \mapsto aca^{-1}$, with $n = k = 3$, $d = 1$, $r = 2$. Note that $r(\hat{\alpha}) = 2$, but the contribution of α is $r_0(\alpha) = (\mathrm{rk}\,\mathrm{Fix}\,\alpha - 1)^+ = 1$, as $\mathrm{Fix}\,\alpha$ has rank 2. The other non-zero contribution is from $\alpha' = i_{a^{-1}} \circ \alpha$, whose fixed subgroup also has rank 2. The point of the next theorem is that this is a general phenomenon: if $\Phi \in \mathrm{Out}\,(F_n)$ satisfies $4e + 2p + 2r + k = 3n - 2$, and $d > 0$, then r has to be carried by at least two isogredience classes (no representative α of Φ satisfies $\mathrm{rk}\,\mathrm{Fix}\,\alpha = r + 1$).

Theorem 4.4. *Given any $\alpha \in \mathrm{Aut}\,(F_n)$, we have*

$$4e + 2d + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k \leq 3n + 1.$$

Note that Theorem 4.3, together with the inequalities $\mathrm{rk}\,\mathrm{Fix}\,\alpha \leq r + 1$ and $d \leq p + 1$, implies $4e + 2d + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k \leq 3n + 2$.

Proof. It suffices to show

$$4e + 2p + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k = 3n \implies d = 0, \tag{*}$$

since any α satisfying $4e + 2d + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k = 3n + 2$ must satisfy $d = p + 1$ and therefore $4e + 2p + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k = 3n$. The proof of $(*)$ is by induction on n.

Consider α satisfying $4e + 2p + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k = 3n$. Note that $\mathrm{Fix}\,\alpha$ must have rank $r + 1$. We argue by way of contradiction, assuming $d > 0$. By Remark 2.2, we may also assume $r > 0$ (we are free to replace α by a power since e, p, k do not change and $\mathrm{rk}\,\mathrm{Fix}\,\alpha$ may only increase). In particular, $\mathrm{Fix}\,\alpha$ has rank $r + 1 \geq 2$.

First suppose that α is polynomially growing. Note that in this case $4e + 2p + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k = 3n$ together with $d = 0$ imply that α is the identity, since α has a power which is inner by Lemma 1.7, and $\mathrm{rk}\,\mathrm{Fix}\,\alpha = k = n$.

After raising α to a power, we are in the situation of Definition 1.1. The representative of $\hat{\alpha}$ introduced in Definition 1.1 is not necessarily equal to α, so we denote it by β. As usual, we distinguish two cases.

- First suppose that there is a decomposition $F_n = G_1 * G_2$ such that β leaves each G_i invariant. The main difficulty is to show that we may assume $\beta = \alpha$.

Consider the action of F_n on the simplicial tree T_0 associated to the free product $G_1 * G_2$. This action is $\hat{\alpha}$-invariant, and there is an isometry H of T_0 representing α in the sense that $\alpha(g)H = Hg$ for all $g \in F_n$ (see [9]). We have to show that H fixes an edge e: if it does, the stabilizers G'_1, G'_2 of the endpoints of e are α-invariant, and α preserves the free product decomposition $F_n = G'_1 * G'_2$.

Since $\mathrm{Fix}\,\alpha$ has rank ≥ 2, the map H has a fixed point [9, 1.1]. The stabilizer of this point is α-invariant, so we may assume $\alpha(G_1) = G_1$ (possibly exchanging the roles of G_1 and G_2 and replacing α by an isogredient automorphism). By an argument given in the proof of 4.3, α and β induce the same outer automorphism Φ_1 of G_1 because G_1 equals its normalizer. The numbers k_i, p_i, r_i used below refer to Φ_1 and to $\Phi_2 = \widehat{\beta_{|G_2}}$

First suppose $\mathrm{Fix}\,\alpha \subset G_1$ (this holds in particular if H has a unique fixed point). Then

$$\begin{aligned} 3n &= 2\mathrm{rk}\,\mathrm{Fix}\,\alpha + k + 2p \\ &\leq 2\mathrm{rk}\,\mathrm{Fix}\,\alpha_{|G_1} + k_1 + k_2 + 2p_1 + 2p_2 + 2 \\ &\leq 3n_1 + 3n_2 - 2 + 2 \\ &= 3n, \end{aligned}$$

using Lemma 2.4 and Theorem 4.3.

All inequalities must be equalities. In particular $d(\Phi_1) = 0$ because $(*)$ is true on G_1 by induction, and as pointed out earlier this implies that α is the identity on G_1. Also $2p = 2p_1 + 2p_2 + 2$, so $d \geq 2$. But $r_2 = 0$, because $r_1 + r_2 \leq r = \operatorname{rk} \operatorname{Fix} \alpha - 1 = r_1$ since $\operatorname{Fix} \alpha = G_1$. Remark 2.2 implies $d_2 = 0$, contradicting Lemma 2.4.

This proves that $\operatorname{Fix} \alpha \subset G_1$ cannot hold. Therefore H fixes an edge, so we may indeed assume $\alpha = \alpha_1 * \alpha_2$. Then $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha_1 * \operatorname{Fix} \alpha_2$. Since we have ruled out $\operatorname{Fix} \alpha \subset G_i$, both $\operatorname{Fix} \alpha_1$ and $\operatorname{Fix} \alpha_2$ are nontrivial. As in the proof of Theorem 4.2, this implies $p = \max(p_1, p_2)$.

We now write:

$$\begin{aligned} 3n &= 2\operatorname{rk} \operatorname{Fix} \alpha + k + 2p \\ &\leq 2\operatorname{rk} \operatorname{Fix} \alpha_1 + 2\operatorname{rk} \operatorname{Fix} \alpha_2 + k_1 + k_2 + 2p_1 + 2p_2 \\ &\leq 3n_1 + 3n_2 \\ &\leq 3n. \end{aligned}$$

Applying $(*)$ to α_1 and α_2 (using the induction hypothesis), we find that α_1 and α_2 are the identity, so α is the identity.

• Now consider a decomposition $F_n = G_1 * \langle t \rangle$ as in Definition 1.1, with β representing $\hat{\alpha}$ such that $\beta(G_1) = G_1$ and $\beta(t) = tu$. The numbers k_1, p_1 refer to $\widehat{\beta_{|G_1}}$. We have shown the inequality $p + r \leq p_1 + r_1 + 1$ in the proof of Proposition 4.2. Using Theorem 4.3, we deduce

$$\begin{aligned} 3n - 2 &= 2r + k + 2p \\ &\leq 2r_1 + k_1 + 2p_1 + 2 + k - k_1 \\ &\leq 3n_1 - 2 + 2 + k - k_1 \\ &\leq 3n - 3 + k - k_1, \end{aligned}$$

showing $k \geq k_1 + 1$. By Lemma 2.5, some decomposition $F_n = G_1 * \langle ta \rangle$ is invariant under a power of β, and we reduce to the previous case.

The proof of Theorem 4.4 is now complete for polynomially growing automorphisms. In the general case, we consider the invariant **R**-tree T as in 1.3.

All elements growing polynomially under α are contained in an α-invariant stabilizer, say G_1. Let $\alpha_1 = \alpha_{|G_1}$, so that $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha_1$ and $r(\hat{\alpha}) = r(\hat{\alpha}_1)$. All Φ_i's with $i \geq 2$ have $r_i = 0$ by Lemma 2.6, hence $d_i = 0$ by Remark 2.2. Since $d = \max d_i$, it suffices to show $d_1 = 0$.

Writing

$$\begin{aligned} 4e + 2\operatorname{rk} \operatorname{Fix} \alpha + k + 2p &\leq 4 + 4\sum e_i + 2\operatorname{rk} \operatorname{Fix} \alpha_1 + k - \sum k_i + \sum k_i + 2p_1 \\ &\leq (4e_1 + 2\operatorname{rk} \operatorname{Fix} \alpha_1 + k_1 + 2p_1) + \sum_{i>1} (4e_i + k_i) + 4 + k - \sum_{i\geq 1} k_i \\ &\leq (4e_1 + 2\operatorname{rk} \operatorname{Fix} \alpha_1 + k_1 + 2p_1) + \sum_{i>1} (3n_i - 2) + 4 + k - \sum_{i\geq 1} k_i \\ &\leq (4e_1 + 2\operatorname{rk} \operatorname{Fix} \alpha_1 + k_1 + 2p_1) + 3n - 3n_1, \end{aligned}$$

we get $4e_1 + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha_1 + k_1 + 2p_1 = 3n_1$, and $d_1 = 0$ by the induction hypothesis. $\square$

By Corollary 1.6, any Φ with $d > 0$ has a power Φ^q represented by an automorphism α with $\mathrm{rk}\,\mathrm{Fix}\,\alpha \geq 2$. In particular, Φ^q satisfies $d \leq p + r$. We thus get from Theorems 4.3 and 4.4:

Corollary 4.5. *Given $\Phi \in \mathrm{Out}\,(F_n)$, we have*

$$\begin{aligned} e + d &\leq n - 1 \\ 4e + 2d &\leq 3n - 2 \\ 4e + 2d &\leq 3n - 3 \quad \textit{if } d > 0. \end{aligned}$$

$\square$

As is easily checked, this corollary is equivalent to saying that (e, d) belongs to the closed quadrilateral with vertices $(0,0), (0, n-1), (\frac{n-1}{2}, \frac{n-1}{2}), (\frac{3n-2}{4}, 0)$ pictured on Figure 1.

We also get:

Corollary 4.6. *Given $\alpha \in \mathrm{Aut}\,(F_n)$, we have*

$$\begin{aligned} e + (d-1)^+ + \mathrm{rk}\,\mathrm{Fix}\,\alpha &\leq n \\ 4e + 2d + 2\mathrm{rk}\,\mathrm{Fix}\,\alpha &\leq 3n + 1 \qquad (\leq 3n \textit{ if } d = 0). \end{aligned}$$

$\square$

We finally prove:

Theorem 4.7. *If $\Phi \in \mathrm{Out}\,(F_n)$ is not polynomially growing, one has $2s + p + r \leq n - 2$. In particular, $s \leq \dfrac{n}{2} - 1$.*

It follows from the appendix that $m \leq s$ if a conjugacy class grows like $\lambda^p p^m$, with $\lambda > 1$, under iteration of Φ. The optimality of $s \leq \frac{n}{2} - 1$ will be shown in the next section.

Proof. The result is true if $s = 0$, by Theorem 4.3, so we assume $s > 0$. We argue by induction on n, using the invariant $\mathbf{R}$-tree T as in 1.3. Since $s > 0$, there is at least one $i_0 \in \{1, \dots, b\}$ such that Φ_{i_0} is not polynomially growing. By Lemma 2.6, there is such an i_0 with $s \leq 1 + s_{i_0}$.

We then get

$$2s + p + r \leq 2s_{i_0} + 2 + \sum_{i=1}^{b} p_i + \sum_{i=1}^{b} r_i \leq n_{i_0} + \sum_{i \neq i_0} (n_i - 1) = 1 + \sum_{i=1}^{b} (n_i - 1)$$

by the induction hypothesis and Theorem 4.3. Since $s > 0$, Proposition 3.1 yields $\sum_{i=1}^{b} (n_i - 1) < n - 2$ and the theorem is proved. $\square$

Corollary 4.8. *If $\alpha \in \mathrm{Aut}\,(F_n)$ is not polynomially growing, one has $2s + d + \mathrm{rk}\,\mathrm{Fix}\,\alpha \leq n$ and $2s + d \leq n - 2$.*

Proof. The first inequality is clear. The second follows from Remark 2.2. $\square$

5. Examples

We give examples, and we show that the inequalities of Corollaries 4.5 and 4.6 are optimal.

Automorphisms of F_2.

Any $\Phi \in \mathrm{Out}\,(F_2)$ is induced by a homeomorphism of a punctured torus. Some power of Φ is either the identity, or a Dehn twist, or a pseudo-Anosov map.

The simplest pseudo-Anosov automorphism is $a \mapsto ab$, $b \mapsto a$. For future reference, we note that its square τ, which sends a to aba and b to ba, fixes the commutator $[a,b] = aba^{-1}b^{-1}$.

Geometric automorphisms.

An automorphism $\Phi \in \mathrm{Out}\,(F_n)$ is geometric if it is induced by a homeomorphism of a compact surface Σ with fundamental group F_n. For Φ geometric, it follows from Nielsen-Thurston theory that the growth of any non-periodic conjugacy class under iteration of Φ (always equivalent to some $\lambda^p p^m$) is either linear or purely exponential: $(\lambda, m) = (1,1)$, or $\lambda > 1$ and $m = 0$.

We now construct a geometric automorphism Φ_n of F_n with e equal to the maximal value $e_n = \left[\frac{3n-2}{4}\right]$. It has a representative $\varphi_n \in \mathrm{Aut}\,(F_n)$ with non-trivial fixed subgroup.

Write $n = 4\ell + 3 + \delta$ with $0 \leq \delta \leq 3$ (for $n = 2$, we take $\Phi_2 = \hat{\tau}$). Construct a compact surface Σ_n with fundamental group F_n by gluing $2\ell + \delta$ once-punctured tori and $\ell + 1$ four-punctured spheres (see Figure 2 for a picture with $\delta = 3$). The number of subsurfaces is $e_n = 3\ell + \delta + 1 = \left[\frac{3n-2}{4}\right]$. Consider an orientation-preserving homeomorphism of Σ_n inducing a pseudo-Anosov map on each of the subsurfaces. The induced automorphism $\Phi_n \in \mathrm{Out}\,(F_n)$ satisfies $e(\Phi_n) = e_n$.

If all pseudo-Anosov maps used in the construction have distinct dilation factors λ_i, then Φ_n has e_n different exponential growth types $(\lambda_i, 0)$.

This example explains the appearance of four-punctured spheres in the proof of Proposition 3.1.

Remark 5.1. If $\Phi \in \mathrm{Out}\,(F_n)$ is induced by a homeomorphism of a compact orientable surface of genus g with b boundary components, one has $n = 2g + b - 1$ and $k \geq b - 1$, so Theorem 4.1 yields $e \leq \frac{3g+b-2}{2}$. It is easy to see that this bound is optimal.

Nested laminations.

The inequality $s \leq \dfrac{n}{2} - 1$ of Theorem 4.7 is an equality for the automorphism α of $F_{2\ell}$ defined by:

$$\begin{cases} a_1 \mapsto a_1 b_1 \\ b_1 \mapsto a_1 \\ a_2 \mapsto a_2 b_2 a_1 \\ b_2 \mapsto a_2 \\ \qquad \vdots \\ a_\ell \mapsto a_\ell b_\ell a_{\ell-1} \\ b_\ell \mapsto a_\ell . \end{cases}$$

The length of $\alpha^p(a_\ell)$ (and of its conjugacy class) is equivalent to $p^{\ell-1}\lambda^p$, with λ the Perron-Frobenius eigenvalue of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Polynomial growth.

Let $n \geq 2$. For the automorphism α_n of F_n defined by $\alpha_n(a_1) = a_1$ and $\alpha_n(a_i) = a_i a_{i-1}$ for $2 \leq i \leq n$, both the element a_i and the conjugacy class $\bar{a}_i$ grow polynomially with degree $i-1$. In particular, $d(\hat{\alpha}_n)$ equals the maximal value $n-1$ (so $\hat{\alpha}$ is not geometric for $n \geq 3$). The rank of $\operatorname{Fix} \alpha_n$ is 2 (it is generated by a_1 and $a_2 a_1 a_2^{-1}$).

We need a slightly different example, with every generator but one mapped to a conjugate. We write x^y for yxy^{-1}.

Lemma 5.2. *Let $\ell \geq 1$. For the automorphism β_ℓ of $F_{\ell+2} = \langle a, a_0, a_1, \ldots, a_\ell \rangle$ defined by :*

$$\begin{cases} a \mapsto a \\ a_0 \mapsto a_0 a \\ a_1 \mapsto {a_1}^{a_0 a} \\ a_2 \mapsto {a_2}^{a_1 a} \\ \qquad \vdots \\ a_\ell \mapsto {a_\ell}^{a_{\ell-1} a}, \end{cases}$$

the conjugacy class of aa_ℓ grows with degree $\ell + 1$. If one adds a generator t with $t \mapsto ta_\ell a$, the class of t grows with degree $\ell + 2$.

Remark. If an automorphism of F_n maps every generator to a conjugate, no conjugacy class grows polynomially with degree $n-1$. This follows from Theorem 4.4 and Corollary 1.6.

Proof. One first shows by induction on $i \geq 1$ that no cancellation occurs when computing iterates $\beta_\ell^p(a_i)$, because the initial letter of $\beta_\ell^p(a_i)$ is a_{i-p} for $p \leq i-1$, and a_0 for $p \geq i$ (and the final letter is the inverse of the initial one). The length

of the (non cyclically reduced word) $\beta_\ell^p(a_\ell)$ is the ℓ^1-norm of the vector

$$\begin{pmatrix} n_a \\ n_0 \\ n_1 \\ \vdots \\ n_\ell \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & \dots & 2 \\ & 1 & 2 & & \\ & & \ddots & \ddots & \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix}^p \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

It grows with degree $\ell+1$. The word $\beta_\ell^p(aa_\ell) = a\beta_\ell^p(a_\ell)$ is cyclically reduced, so the class of aa_ℓ grows with degree $\ell+1$.

The p-th iterate of the new generator t is a cyclically reduced word containing $a_\ell, \beta_\ell(a_\ell), \dots, \beta_\ell^{p-1}(a_\ell)$ as disjoint subwords, so the class of t grows with degree $\ell+2$. $\square$

Mixed growth.

For $n \geq 3$, we construct an automorphism θ_n of F_n with e and d as close as possible to $(n-1)/2$. Its fixed subgroup has rank 2.

First assume n odd, and write $n = 2\ell+3$. Consider $F_n = \langle a, b, a_0, a_1, b_1, \dots, a_\ell, b_\ell \rangle$. Let $u = [a,b]$ and $u_i = [a_i, b_i]$. Recall that $\tau : (a,b) \mapsto (aba, ba)$ is an exponentially growing automorphism fixing $u = [a,b]$. Define $\theta_n \in \mathrm{Aut}\,(F_n)$ by:

$$\begin{cases} a \mapsto aba \\ b \mapsto ba \\ a_0 \mapsto a_0 u \\ a_1 \mapsto (a_1 b_1 a_1)^{a_0 u} \\ b_1 \mapsto (b_1 a_1)^{a_0 u} \\ a_2 \mapsto (a_2 b_2 a_2)^{u_1 u} \\ b_2 \mapsto (b_2 a_2)^{u_1 u} \\ \quad \vdots \\ a_\ell \mapsto (a_\ell b_\ell a_\ell)^{u_{\ell-1} u} \\ b_\ell \mapsto (b_\ell a_\ell)^{u_{\ell-1} u} \end{cases}$$

Geometrically, θ_n is represented by a homotopy equivalence ψ on a 2-complex X_n built as follows. Take disjoint punctured tori $T, T_1, \dots, T_\ell$, with points $v, v_1, \dots, v_\ell$ on the boundary. Glue a circle to v, and add edges vv_i. The map ψ induces a pseudo-Anosov homeomorphism on each punctured torus.

The automorphism θ_n has $\ell+1$ exponential strata. Furthermore, consider the subgroup P_0 generated by $u, a_0, u_1, \dots, u_\ell$ (it is the fundamental group of the 1-complex obtained from X_n by removing the interior of each torus). It is θ_n-invariant, and the restriction of θ_n to P_0 is β_ℓ. Thus $\hat{\theta}_n$ satisfies $e = d = \ell+1 = \frac{n-1}{2}$ by Lemma 5.2. Both inequalities of Corollary 4.5 are equalities.

If n is even, we write $n = 2\ell + 4$ and we add a generator t mapped to $tu_\ell u$. We get an automorphism with $e = \frac{n}{2} - 1$ and $d = \frac{n}{2}$ by Lemma 5.2.

The inequality $e + d \leq n - 1$ of Corollary 4.5 is an equality.

These examples have only one exponential growth type $(\lambda, 0)$, with λ the Perron-Frobenius eigenvalue of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It is easy to modify the construction so that there are e distinct exponential growth types $(\lambda^{i+1}, 0)$, $0 \leq i \leq \ell$, by using τ^{i+1} rather than τ when defining the image of a_i and b_i.

Optimality.

We can now show:

Theorem 5.3. *Given (ε, δ) belonging to the closed quadrilateral with vertices $(0,0), (0, n-1), (\frac{n-1}{2}, \frac{n-1}{2}), (\frac{3n-2}{4}, 0)$, there exists $\alpha \in \operatorname{Aut}(F_n)$ such that:*

1. *Any improved relative train track map representing a power of $\hat{\alpha}$ has ε exponential strata. Equivalently, there are ε attracting laminations.*
2. *There are ε distinct exponential growth rates $(\lambda_i, 0)$ with $\lambda_i > 1$.*
3. *There is a conjugacy class whose growth is polynomial of degree δ.*
4. *The rank of $\operatorname{Fix} \alpha$ is the maximal value ρ_0 permitted by the inequalities of Corollary 4.6.*

Proof. We write ρ for $\operatorname{rk} \operatorname{Fix} \alpha$. The inequalities of Corollary 4.6 are

$$\begin{aligned} \varepsilon + (\delta - 1)^+ + \rho &\leq n \\ 4\varepsilon + 2\delta + 2\rho &\leq 3n + 1 \qquad (\leq 3n \text{ if } \delta = 0). \end{aligned}$$

Since (ε, δ) is in the quadrilateral, the maximal value ρ_0 is always ≥ 1. Which of the two inequalities is the limiting one depends on the position of ε with respect to $n/2$ if $\delta = 0$, with respect to $(n-1)/2$ if $\delta > 0$.

We shall construct α satisfying (1), (3), (4), using the automorphisms $\tau, \varphi_n, \alpha_n, \theta_n$ introduced above. As in the construction of θ_n, one achieves (2) by using varying powers of τ.

We write I_ℓ for the identity automorphism of F_ℓ.

• If $\varepsilon = 0$ and $\delta > 0$, we define $\alpha = \alpha_{\delta+1} * I_{n-\delta-1}$, with $\alpha_{\delta+1} \in \operatorname{Aut}(F_{\delta+1})$ defined above.

• Suppose $\delta = 0$. The inequalities are $\varepsilon + \rho \leq n$ and $4\varepsilon + 2\rho \leq 3n$.

If $\varepsilon \leq \frac{n}{2}$, we write F_n as the free product of $F_{n-2\varepsilon}$ and ε copies of F_2, and we define $\alpha = I_{n-2\varepsilon} * \tau * \cdots * \tau$. Then $\rho = n - 2\varepsilon + \varepsilon = \rho_0$.

If $\varepsilon > \frac{n}{2}$, first suppose n is even, say $n = 2\ell$. The required value of ρ is $\rho_0 = \frac{3n-4\varepsilon}{2} = 3\ell - 2\varepsilon$. Write F_n as the free product of $F_{4\varepsilon-4\ell+2}$ with $3\ell - 2\varepsilon - 1$ copies of F_2 and define $\alpha = \varphi_{4\varepsilon-4\ell+2} * \tau * \cdots * \tau$. The number of exponential strata is $3\varepsilon - 3\ell + 1 + 3\ell - 2\varepsilon - 1 = \varepsilon$, and $\rho = 1 + 3\ell - 2\varepsilon - 1 = \rho_0$. If $n = 2l + 1$, we take the free product of $\varphi_{4\varepsilon-4\ell+1}$ with $3\ell - 2\varepsilon$ copies of τ. Then $e = 3\varepsilon - 3\ell + 3\ell - 2\varepsilon = \varepsilon$ and $\rho = 1 + 3\ell - 2\varepsilon = \rho_0$.

We now suppose $\delta, \varepsilon \geq 1$. This implies $n \geq 3$ and $\rho_0 \geq 2$.

• Suppose δ, $\varepsilon \geq 1$, and $\varepsilon \leq (n-1)/2$. Then $\rho_0 = n - \varepsilon - \delta + 1$.

The construction uses auxiliary parameters $w, x, y, z \geq 0$, to be determined in terms of δ and ε.

Starting with θ_{2w+3}, for which $d = e = w + 1$ and $\rho = 2$, we add x generators so as to obtain an automorphism of F_{2w+3+x} with $e = w + 1$ and $d = w + 1 + x$ (map the first generator t_1 to $t_1 u_\ell u$ as in Lemma 5.2, then t_i to $t_i t_{i-1}$ as in the definition of α_n). We then take the free product with I_y and with z copies of τ.

We get an automorphism of a group of rank $2w+3+x+y+2z$, with $d = w+1+x$, $e = w + 1 + z$, and $\rho = 2 + y + z$. We must therefore find $w, x, y, z \geq 0$ satisfying

$$\begin{aligned} w + 1 + x &= \delta \\ w + 1 + z &= \varepsilon \\ 2 + y + z &= \rho_0 = n - \varepsilon - \delta + 1 \\ 2w + 3 + x + y + 2z &= n. \end{aligned}$$

Note that the last equation is the sum of the others. If we know w, we get x, y, z by

$$\begin{aligned} x &= \delta - w - 1 \\ y &= \rho_0 - \varepsilon + w - 1 \\ z &= \varepsilon - w - 1. \end{aligned}$$

We have to choose w so that w, x, y, z are non-negative. This is equivalent to $w \geq 0$, $w \geq \varepsilon - \rho_0 + 1$, $w \leq \delta - 1$, $w \leq \varepsilon - 1$. Since δ and ε are ≥ 1, and $\rho_0 \geq 2$, we only need to check $\varepsilon - \rho_0 + 1 \leq \delta - 1$. This holds because $\rho_0 = n - \varepsilon - \delta + 1$ and $\varepsilon \leq (n-1)/2$.

• Finally, suppose δ, $\varepsilon \geq 1$, and $\varepsilon > (n-1)/2$. Then $\rho_0 = \left\lceil \dfrac{3n + 1 - 4\varepsilon - 2\delta}{2} \right\rceil$.

First assume that n is odd. We use three parameters w, x, z. We first combine φ_{4x+2} with θ_{2w+3}, in the following sense. We consider $F_{4x+2w+3} = F_{4x+2} * \langle a_0, a_1, b_1, \ldots, a_w, b_w \rangle$. Let u be a generator for the fixed subgroup of φ_{4x+2}. We define an automorphism of $F_{4x+2w+3}$ as being equal to φ_{4x+2} on the first factor and mapping $a_0, a_1, b_1, \ldots, a_w, b_w$ by the same formulas as in the definition of θ_{2w+3}. This automorphism has $e = 3x + 1 + w$, $d = w + 1$, and $\rho = 2$. We then take the free product with z copies of τ, so as to increase e and ρ by z.

We now have to solve:

$$\begin{aligned} w + 1 &= \delta \\ 3x + 1 + w + z &= \varepsilon \\ 2 + z &= \rho_0 \\ 4x + 2w + 3 + 2z &= n. \end{aligned}$$

We have assumed n to be odd. Setting $n = 2\ell + 1$, we have $\rho_0 = 3\ell + 2 - 2\varepsilon - \delta$. It is easy to check that $w = \delta - 1$, $z = \rho_0 - 2$, $x = \varepsilon - \ell$ is a non-negative solution.

For $n = 2\ell + 2$, we use the same construction with φ_{4x+1} rather than φ_{4x+2}, defining w, z, x by the exact same formulas (now $\rho_0 = 3\ell + 3 - 2\varepsilon - \delta$, the second equation is $3x + w + z = \varepsilon$, and the fourth one is $4x + 2w + 2 + 2z = n$). $\square$

6. Appendix: Growth

More on train tracks.

Let $f : G \to G$ be an improved relative train track map. We recall some more definitions from [1].

We write $f_\sharp(\gamma)$ for the tightened image of γ (the reduced path homotopic to $f(\gamma)$ rel. endpoints). A decomposition $\gamma = \gamma_1 \dots \gamma_q$ is a splitting if $f^p_\sharp(\gamma) = f^p_\sharp(\gamma_1) \dots f^p_\sharp(\gamma_q)$ for all $p \geq 1$ (i.e. there is no cancellation between $f^p_\sharp(\gamma_j)$ and $f^p_\sharp(\gamma_{j+1})$). The subpaths γ_j are the pieces of the spitting, and we say that γ splits over each γ_j.

If H_i is an NEG stratum, it consists of a single edge e_i, and $f(e_i)$ splits as $e_i.u_i$ with u_i of height $< i$. If e is an edge in an exponential stratum H_i, then $f(e)$ has a splitting whose pieces are edges of H_i or paths of height $< i$.

Let H_i be exponential. By aperiodicity, every edge of H_i appears in $f(e)$, for e any edge of H_i. Any subpath of $f^p_\sharp(e)$ is i-legal (for the purposes of this appendix, this may be taken as the definition of i-legal). We call Δ_i the (finite) collection of maximal subpaths of height $< i$ which appear in $f(e)$, for e an edge of H_i. If $\delta \in \Delta_i$, then no $f^p_\sharp(\delta)$ is a point. It follows from bounded cancellation that there exists a constant K_i such that, if a path γ contains an i-legal subpath with more than K_i edges in H_i, then γ splits over an edge of H_i (see 4.2 in [1]).

As mentioned earlier, there is a bijection between the set of attracting laminations of the automorphism represented by f and the set of exponential strata of f. A bi-infinite path γ in G is a leaf of the lamination associated to H_i if and only if any finite subpath is contained in some $f^p_\sharp(e)$, with e an edge of H_i.

Lemma 6.1. *Let f be an improved relative train track map. Let H_i, H_j be exponential strata, with associated laminations Λ_i, Λ_j. The following are equivalent:*

(1) $\Lambda_j \subset \Lambda_i$.
(2) *There exist edges e_i, e_j of H_i, H_j, and $p \geq 1$, such that $f^p_\sharp(e_i)$ splits over e_j.*
(3) *Given edges e_i, e_j of H_i, H_j, there exists $p \geq 1$ such that $f^p_\sharp(e_i)$ splits over e_j.*

Proof. (2) $\iff$ (3) follows from aperiodicity. If $f^p_\sharp(e_i)$ splits over e_j, every $f^q_\sharp(e_j)$ is contained in $f^{p+q}_\sharp(e_i)$, so every leaf of Λ_j is a leaf of Λ_i. Conversely, if $\Lambda_j \subset \Lambda_i$, let γ_0 be a segment in a leaf of Λ_j. It is contained in some $f^p_\sharp(e_i)$, and by bounded cancellation (see above) $f^p_\sharp(e_i)$ splits over some edge of H_j contained in γ_0 if γ_0 is long enough. $\square$

Remark. In particular, $j \leq i$ if $\Lambda_j \subset \Lambda_i$. But the total order on the set of exponential strata of f defined by $H_i \leq H_j$ if $i \leq j$ does not have an intrinsic meaning.

The Perron-Frobenius eigenvalue λ_i of H_i is called the expansion factor of Λ_i (for the automorphism represented by f).

Given $\Phi \in \mathrm{Out}\,(F_n)$, it is only true that some power Φ^q is represented by $f : G \to G$ as above. The attracting laminations of Φ are those of Φ^q, and we define the expansion factor of Λ for Φ as $\lambda^{1/q}$, where λ is the expansion factor of Λ for Φ^q.

Growth types.

Given $\lambda \geq 1$ and $m \in \mathbf{N}$, we say that a conjugacy class $\overline{g}$ grows like $\lambda^p p^m$ under Φ, or has growth type (λ, m), if the length $|\Phi^p(\overline{g})|$ grows like $\lambda^p p^m$ in the sense that there exist constants $C_1, C_2 > 0$ with $C_1 \lambda^p p^m \leq |\Phi^p(\overline{g})| \leq C_2 \lambda^p p^m$ for all $p \geq 1$. The set of growth types is ordered lexicographically, so that $(\lambda, m) \leq (\lambda', m')$ if $\lambda^p p^m \leq \lambda'^p p^{m'}$ as $p \to \infty$.

We define similarly the growth type of an element $g \in F_n$ under $\alpha \in \mathrm{Aut}\,(F_n)$, and of an edge-path (or a loop) γ in G under a relative train track map $f : G \to G$ by considering the simplicial length $|f_\sharp^p(\gamma)|$.

If g or $\overline{g}$ grows like $\lambda^p p^m$ under some positive power α^q or Φ^q, it grows like $(\lambda^{1/q})^p p^m$ under α or Φ. This allows us to replace an automorphism by a power whenever convenient.

We consider the set of attracting laminations of Φ, ordered by inclusion. Each attracting lamination Λ has an expansion factor $\lambda_0 > 1$ (see above). From this data, we shall now associate to each Λ a growth type $c = (\lambda, m)$, with $\lambda > 1$ and $m \in \mathbf{N}$. The definition will ensure that, if Λ is associated to an exponential stratum H_i of $f : G \to G$ representing Φ, and e is an edge of H_i, then the length of $f_\sharp^p(e)$ grows like $\lambda^p p^m$ (see Proposition 6.4).

The definition is by induction on the number of laminations contained in Λ, using the following rules. Let λ_0 be the expansion factor of Λ. If Λ is minimal (for inclusion), then $c = (\lambda_0, 0)$. If not, let (λ', m') be the maximum growth type for $\Lambda' \subsetneq \Lambda$. If $\lambda' < \lambda_0$, then $c = (\lambda_0, 0)$. If $\lambda' > \lambda_0$, then $c = (\lambda', m')$. If $\lambda' = \lambda_0$, then $c = (\lambda_0, m' + 1)$. In all cases, $c(\Lambda) = \max_{\Lambda' \subseteq \Lambda} c(\Lambda')$. Also note that, if $m > 0$, there is a decreasing chain of laminations $\Lambda = \Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_m$, so that $m \leq s$.

Example. Consider the automorphism of F_4 defined by $a \mapsto abaa'$, $b \mapsto ba$, $a' \mapsto a'b'$, $b' \mapsto a'$. There are two attracting laminations Λ and Λ', with $\Lambda' \subset \Lambda$. The expansion factor λ_0 of Λ is the Perron-Frobenius eigenvalue μ of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the expansion factor λ'_0 of Λ' is the eigenvalue ν of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. One has $\mu > \nu$; the growth type of Λ is $(\mu, 0)$, corresponding to the fact that $|\Phi^p(\overline{a})|$ grows like μ^p. For the automorphism $a \mapsto aba'$, $b \mapsto a$, $a' \mapsto a'b'a'$, $b' \mapsto a'b'$, one still has two laminations $\Lambda' \subset \Lambda$, but now $\lambda_0 = \nu$ and $\lambda'_0 = \mu$, so $\lambda_0 < \lambda'_0$. The growth type of Λ is $(\mu, 0)$, and $|\Phi^p(\overline{a})|$ grows like μ^p. Finally, consider $a \mapsto aba'$, $b \mapsto a$, $a' \mapsto a'b'$,

$b' \mapsto a'$. One has $\lambda_0 = \lambda'_0 = \nu$. The growth type of Λ is $(\nu, 1)$, and $|\Phi^p(\overline{a})|$ grows like $p\nu^p$.

Theorem 6.2. *Let $\Phi \in \mathrm{Out}\,(F_n)$.*

(1) *Given $g \in F_n$, there exist $\lambda \geq 1$ and $m \in \mathbf{N}$ such that $|\Phi^p(\overline{g})|$ grows like $\lambda^p p^m$.*
(2) *If $|\Phi^p(\overline{g})|$ grows like $\lambda^p p^m$, and $m' < m$, there exists g' such that $|\Phi^p(\overline{g}')|$ grows like $\lambda^p p^{m'}$.*
(3) *Given (λ, m) with $\lambda > 1$, there exists g such that $|\Phi^p(\overline{g})|$ grows like $\lambda^p p^m$ if and only if (λ, m) is the growth type of some attracting lamination of Φ.*

When $\lambda > 1$, the power m which appears in the growth of $|\Phi^p(\overline{g})|$ satisfies $m \leq s \leq n/2 - 1$ by Theorem 4.7. When $\lambda = 1$, the maximum value of m is the number that we have called d. It satisfies $d \leq n - 1$ (see Corollary 4.5).

Before proving Theorem 6.2, we also note:

Corollary 6.3. *Given $\alpha \in \mathrm{Aut}\,(F_n)$ and $g \in F_n$, there exist $\lambda \geq 1$ and $m \in \mathbf{N}$ such that $|\alpha^p(g)|$ grows like $\lambda^p p^m$.*

Proof. Extend α to an automorphism β of F_{n+1} by mapping the new generator t to itself. The growth of g under α is that of the conjugacy class $\overline{tg}$ under $\hat{\beta}$. □

The set of growth types of elements of F_n under α is the same as the set of growth types of conjugacy classes under $\hat{\alpha}$, except that there may be elements with growth p^{d+1} (see Lemma 2.3).

Proof of Theorem 6.2.

The rest of this section is devoted to the proof of Theorem 6.2. See [2, 4, 12] for partial results. Our proof elaborates on an argument due to Bridson-Groves.

After replacing Φ by a power, we may assume that it is represented by an improved relative train track map $f : G \to G$. The heart of the proof is to show that *any edge-path or loop γ in G has a growth type* (λ, m), in the sense that the length of $f^p_\sharp(\gamma)$ is bounded between $C_1\lambda^p p^m$ and $C_2\lambda^p p^m$ for some $C_1, C_2 > 0$.

Recall that we have defined a growth type $c = (\lambda, m)$ for an attracting lamination Λ, hence also for an exponential stratum H_i. We write c_i for the growth type attached to H_i.

Proposition 6.4. *Given γ, let C_γ be the set of all j such that some $f^p_\sharp(\gamma)$ splits over an edge belonging to an exponential stratum H_j. If $C_\gamma = \emptyset$, then γ grows like p^m for some $m \in \mathbf{N}$. Otherwise, the growth type of γ is the maximal c_j, for $j \in C_\gamma$.*

In particular, the growth type of an edge in an exponential stratum H_i is c_i by Lemma 6.1 and the equality $c(\Lambda) = \max_{\Lambda' \subseteq \Lambda} c(\Lambda')$.

The proof of the proposition is by induction on the height of γ. First suppose that γ is a single edge e in an exponential stratum H_i, and the proposition is true for paths of height $< i$. Let $c_i = (\lambda, m)$, and let $\lambda_i \leq \lambda$ be the Perron-Frobenius eigenvalue attached to H_i. We show that $|f^p_\sharp(e)|$ grows like $\lambda^p p^m$.

As mentioned above, $f(e)$ splits over edges of H_i and paths $\delta \in \Delta_i$ (recall that we have defined Δ_i as the finite set of maximal subpaths of height $< i$ which appear in $f(e')$ for e' an edge of H_i). Thus $f_\sharp^p(e)$ splits over edges of H_i and paths of the form $f_\sharp^q(\delta)$ with $q \leq p-1$ and $\delta \in \Delta_i$. Up to constants (which we will not write), the number of edges in $f_\sharp^p(e) \cap H_i$ is λ_i^p, and for given $q \leq p-1$ and $\delta \in \Delta_i$ the number of subpaths $f_\sharp^q(\delta)$ is λ_i^{p-q-1} (as they are created by edges in $f_\sharp^{p-q-1}(e) \cap H_i$).

We first show that the length of $f_\sharp^p(e)$ grows at most like $\lambda^p p^m$. It suffices to show that, for a given δ in the set Δ_i, the sum $\sum_{q=1}^p \lambda_i^{p-q} |f_\sharp^q(\delta)|$ grows at most like $\lambda^p p^m$. If some $f_\sharp^q(\delta)$ splits over an edge in an exponential stratum H_j, then so does some $f_\sharp^{q'}(e)$. By the induction hypothesis, $|f_\sharp^q(\delta)|$ grows either polynomially or with growth type $c_j = (\lambda_j, m_j)$, with $j \in C_\gamma$ and $j < i$, so we have to show that $S_p = \sum_{q=1}^p \lambda_i^{p-q} \lambda_j^q q^{m_j}$ grows at most like $\lambda^p p^m$. But the growth type of a lamination was defined in such a way that this holds, since S_p grows like λ_i^p if $\lambda_j < \lambda_i$, like $\lambda_j^p p^{m_j}$ if $\lambda_j > \lambda_i$, like $\lambda_j^p p^{m_j+1}$ if $\lambda_j = \lambda_i$.

We now show that e grows at least like $\lambda^p p^m$. This is clear if Λ_i is minimal (for inclusion). If $\lambda > \lambda_i$, there is $\Lambda_j \subsetneq \Lambda_i$ with $c_j = c_i$. By Lemma 6.1, some $f_\sharp^q(e)$ splits over an edge e_j of H_j, so e grows as fast as e_j, whose growth type is c_j by induction. The case when $\lambda = \lambda_i$ is harder.

The result is clear if $m = 0$, so assume $m > 0$. Then there is $\Lambda_j \subsetneq \Lambda_i$ with $c_j = (\lambda, m-1)$. Some $f_\sharp^q(e)$ splits over an edge e_j of H_j. This edge is contained in some $f_\sharp^{q'}(\delta)$, with $q' \leq q-1$ and $\delta \in \Delta_i$. The splitting of $f_\sharp^q(e)$ over e_j may not be compatible with the splitting of $f_\sharp^q(e)$ over $f_\sharp^{q'}(\delta)$. But further iterates of e split over long j-legal paths, and by bounded cancellation we may therefore assume (after increasing q) that a subpath $\delta' = f_\sharp^{q'}(\delta)$ of $f_\sharp^q(e)$ splits over an edge of H_j, hence has growth type at least $(\lambda, m-1)$ by induction.

Fix q and δ', and consider $f_\sharp^{qp}(e)$. For each $\ell < p$ it contains (up to constants) at least $\lambda^{q(p-\ell)}$ subpaths $f_\sharp^{q\ell}(\delta')$, each with length $\lambda^{q\ell}(q\ell)^{m-1}$. It follows that the length of $f_\sharp^{qp}(e)$ is bounded below by $\sum_{\ell=1}^{p-1} \lambda^{q(p-\ell)} \lambda^{q\ell} (q\ell)^{m-1}$, hence by $\lambda^{qp}(qp)^m$.

This completes the proof of the induction step for γ an edge in an exponential stratum. In general, we now know that γ grows at least like the maximal c_j. For the upper bound, we use the following fact.

Lemma 6.5 [1, Lemmas 4.1.4, 4.2.6, 5.5.1]. *If γ has height i, there exists p_0 such that $f_\sharp^{p_0}(\gamma)$ has a splitting whose pieces are edges of H_i, paths of height $< i$, Nielsen paths, and exceptional paths.* □

Nielsen paths do not grow, and exceptional paths grow linearly. Paths of height $< i$ grow at most like the maximal c_j by the induction hypothesis. We know how edges in an exponential stratum grow, so there only remains the case when γ is an edge in an NEG stratum. We have $f_\sharp^p(\gamma) = \gamma u f_\sharp(u) f_\sharp^2(u) \dots f_\sharp^{p-1}(u)$ for some path u of height $< i$. By the induction hypothesis, u grows either polynomially or

with growth type c_j with $j \in C_\gamma$. Thus γ grows at the same speed as u if u grows exponentially, like p^{m+1} if u grows like p^m.

The proof of Proposition 6.4 is now complete. To prove the first assertion of Theorem 6.2, we simply observe that the type of growth of a conjugacy class $\overline{g}$ (under Φ) is the same as that of the loop representing $\overline{g}$ in G (under f). If H_i is an exponential stratum, any conjugacy class represented by an i-legal loop meeting H_i has growth type c_i. This shows assertion (3), and assertion (2) when $\lambda > 1$. Assertion (2) in the polynomial case is well-known, but we sketch a proof for completeness. If $\overline{g}$ grows like p^m, the arguments given above imply the existence of a path u which grows like p^{m-1}. This path is a loop by [1], but images $f_\sharp^p(u)$ may fail to be reduced as loops. It is easy to check, however, that for each p one of the loops $f_\sharp^p(u)$, $f_\sharp^{p+1}(u)$, $f_\sharp^p(u)f_\sharp^{p+1}(u)$ is reduced, and assertion (2) follows.

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